GRAPH THEORY - ISOMORPHISM

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A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs. Note that we label the graphs in this chapter mainly for the purpose of referring to them and recognizing them from one another.

Isomorphic Graphs

Two graphs G_1 and G_2 are said to be isomorphic if −

- Their number of components *verticesandedges* are same.
- Their edge connectivity is retained.

Note − In short, out of the two isomorphic graphs, one is a tweaked version of the other. An unlabelled graph also can be thought of as an isomorphic graph.

There exists a function 'f' from vertices of G_1 to vertices of G_2
  [$f: V(G_1) \Rightarrow V(G_2)$], such that
Case (i): *f* is a bijection (both one-one and onto)
Case (ii): *f* preserves adjacency of vertices, i.e., if the edge $\{U, V\} \in G_1$, then the
  edge $\{f(U), f(V)\} \in G_2$, then $G_1 \equiv G_2$.

Note

If $G_1 \equiv G_2$ then −

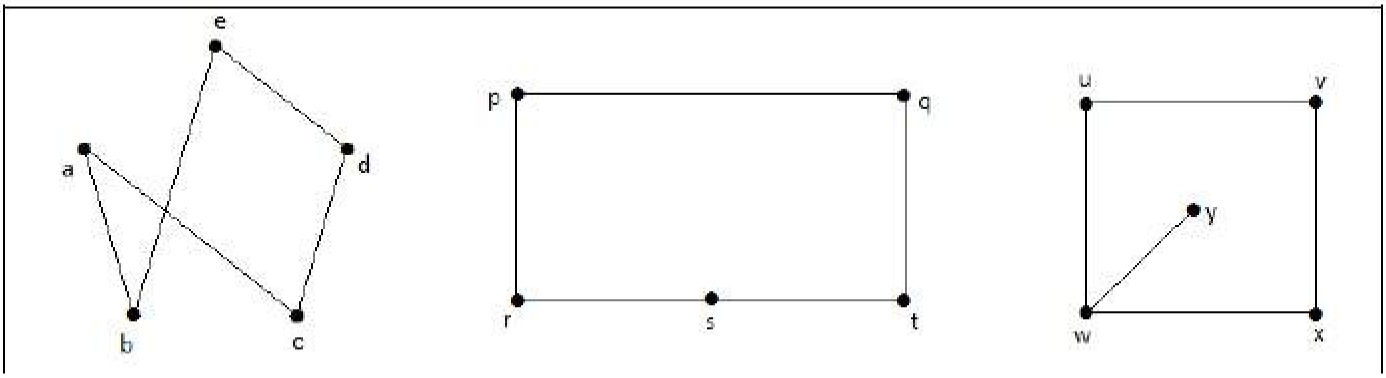
- $|V(G_1)| = |V(G_2)|$
- $|E(G_1)| = |E(G_2)|$
- Degree sequences of G_1 and G_2 are same.
- If the vertices $\{V_1, V_2, .. V_k\}$ form a cycle of length K in G_1, then the vertices $\{f(V_1), f(V_2),\ldots f(V_k)\}$ should form a cycle of length K in G_2.

All the above conditions are necessary for the graphs G_1 and G_2 to be isomorphic, but not sufficient to prove that the graphs are isomorphic.

- $(G_1 \equiv G_2)$ if and only if (−) where G_1 and G_2 are simple graphs.
- $(G_1 \equiv G_2)$ if the adjacency matrices of G_1 and G_2 are same.
- $(G_1 \equiv G_2)$ if and only if the corresponding subgraphs of G_1 and G_2 (obtained by deleting some vertices in G_1 and their images in graph G_2) are isomorphic.

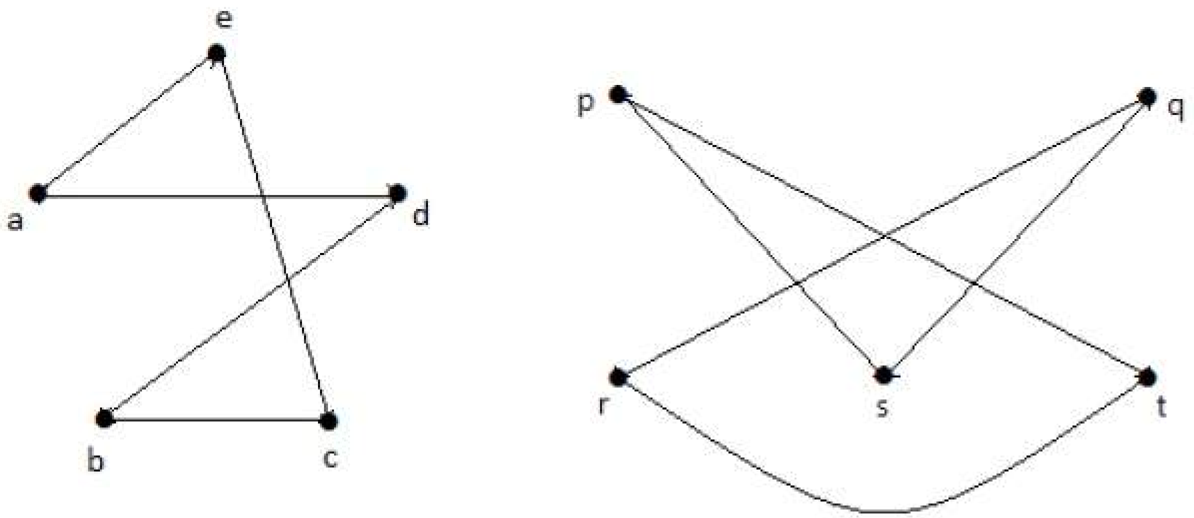
Example

Which of the following graphs are isomorphic?



In the graph G_3, vertex ‘w’ has only degree 3, whereas all the other graph vertices has degree 2. Hence G_3 not isomorphic to G_1 or G_2.

Taking complements of G_1 and G_2, you have −

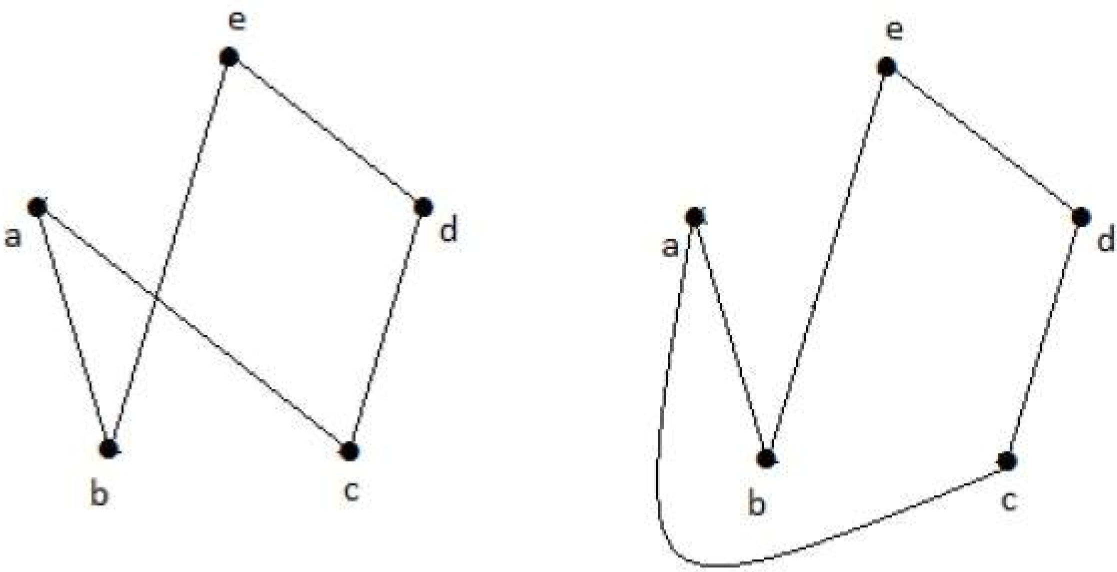


Here, (−), hence ($G_1 \equiv G_2$).

Planar Graphs

A graph ‘G’ is said to be planar if it can be drawn on a plane or a sphere so that no two edges cross each other at a non-vertex point.

Example

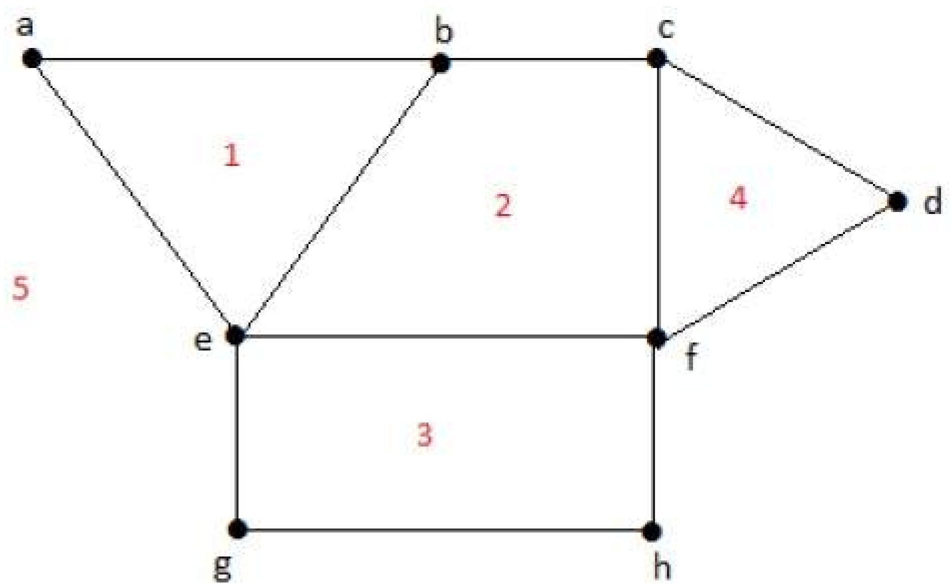


NON - PLANAR GRAPH   PLANAR GRAPH

Regions

Every planar graph divides the plane into connected areas called regions.

Example



Degree of a bounded region **r = deg*r*** = Number of edges enclosing the regions **r**.

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deg(1) = 3
deg(2) = 4
deg(3) = 4
deg(4) = 3
deg(5) = 8
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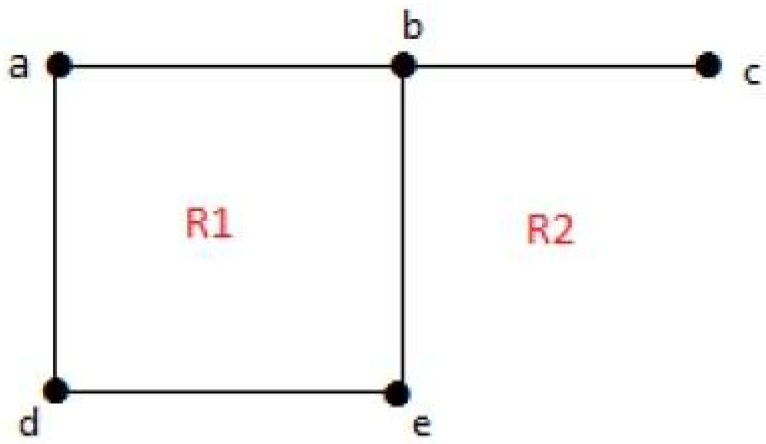

Degree of an unbounded region **r = deg*r*** = Number of edges enclosing the regions **r**.

$deg(R_1) = 4$
$deg(R_2) = 6$

In planar graphs, the following properties hold good −

- **1.** In a planar graph with 'n' vertices, sum of degrees of all the vertices is

$$\sum_{i=1}^{n} deg(V_i) = 2|E|$$

- **2.** According to **Sum of Degrees of Regions** Theorem, in a planar graph with 'n' regions, Sum of degrees of regions is −

$$\sum_{i=1}^{n} deg(r_i) = 2|E|$$

Based on the above theorem, you can draw the following conclusions −

In a planar graph,

- If degree of each region is K, then the sum of degrees of regions is

$$K|R| = 2|E|$$

- If the degree of each region is at least K $\geq K$, then

$$K|R| \leq 2|E|$$

- If the degree of each region is at most K $\leq K$, then

$$K|R| \geq 2|E|$$

Note − Assume that all the regions have same degree.

3. According to **Euler's Formulae** on planar graphs,

- If a graph 'G' is a connected planar, then

$$|V| + |R| = |E| + 2$$

- If a planar graph with 'K' components then

$$|V| + |R| = |E| + K + 1$$

Where, |V| is the number of vertices, |E| is the number of edges, and |R| is the number of regions.

4. Edge Vertex Inequality

If 'G' is a connected planar graph with degree of each region at least 'K' then,

|E| ≤ *k / k − 2* {|v| - 2}

You know, |V| + |R| = |E| + 2

K.|R| ≤ 2|E|

K |E| - |V| + 2 ≤ 2|E|

$K - 2$|E| ≤ K$|V| - 2$

|E| ≤ *k / k − 2* {|v| - 2}

5. If 'G' is a simple connected planar graph, then

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|E| ≤ 3|V| − 6
|R| ≤ 2|V| − 4
```

There exists at least one vertex V ∈ G, such that deg$V$ ≤ 5

6. If 'G' is a simple connected planar graph $withatleast2edges$ **and no triangles, then**

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|E| ≤ {2|V| – 4}
```
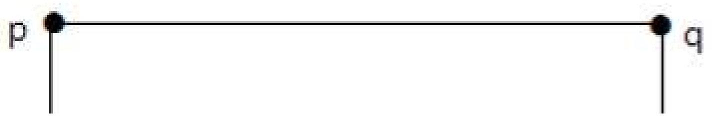

7. Kuratowski's Theorem

A graph 'G' is non-planar if and only if 'G' has a subgraph which is homeomorphic to K_5 or $K_{3,3}$.

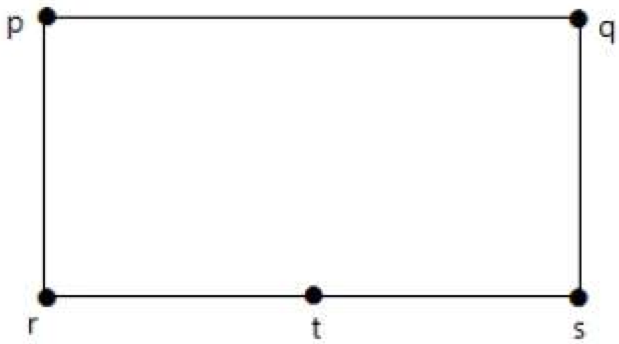
Homomorphism

Two graphs G_1 and G_2 are said to be homomorphic, if each of these graphs can be obtained from the same graph 'G' by dividing some edges of G with more vertices. Take a look at the following example −

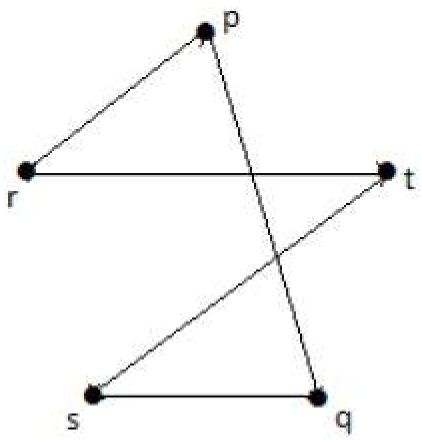


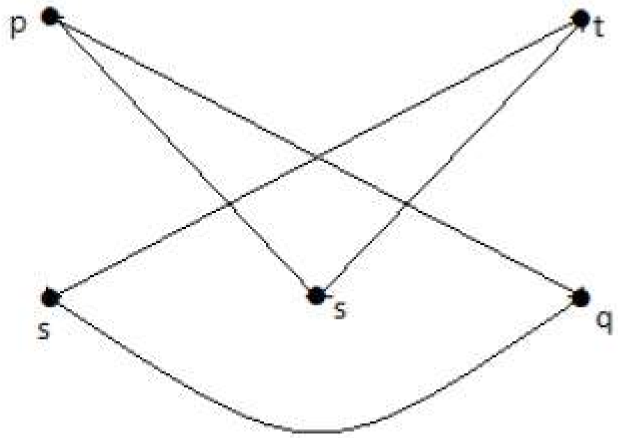
G

Divide the edge ‘rs’ into two edges by adding one vertex.



The graphs shown below are homomorphic to the first graph.





If G_1 is isomorphic to G_2, then G is homeomorphic to G_2 but the converse need not be true.

- Any graph with 4 or less vertices is planar.
- Any graph with 8 or less edges is planar.
- A complete graph K_n is planar if and only if $n \leq 4$.
- The complete bipartite graph $K_{m, n}$ is planar if and only if $m \leq 2$ or $n \leq 2$.
- A simple non-planar graph with minimum number of vertices is the complete graph K_5.
- The simple non-planar graph with minimum number of edges is $K_{3, 3}$.

Polyhedral graph

A simple connected planar graph is called a polyhedral graph if the degree of each vertex is ≥ 3, i.e., $\deg V \geq 3 \; \forall \; V \in G$.

- 3|V| ≤ 2|E|
- 3|R| ≤ 2|E|

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