GRAPH THEORY - TYPES OF GRAPHS

There are various types of graphs depending upon the number of vertices, number of edges, interconnectivity, and their overall structure. We will discuss only a certain few important types of graphs in this chapter.

Null Graph

A **graph having no edges** is called a Null Graph.

Example

In the above graph, there are three vertices named 'a', 'b', and 'c', but there are no edges among them. Hence it is a Null Graph.

Trivial Graph

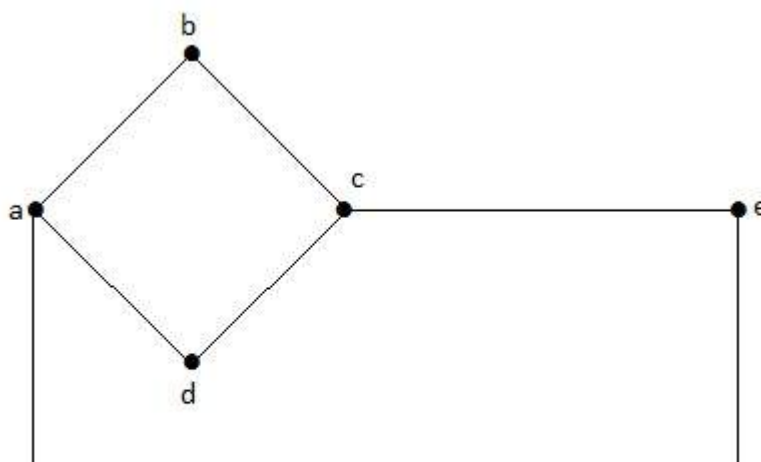A **graph with only one vertex** is called a Trivial Graph.

Example

In the above shown graph, there is only one vertex 'a' with no other edges. Hence it is a Trivial graph.

Non-Directed Graph

A non-directed graph contains edges but the edges are not directed ones.
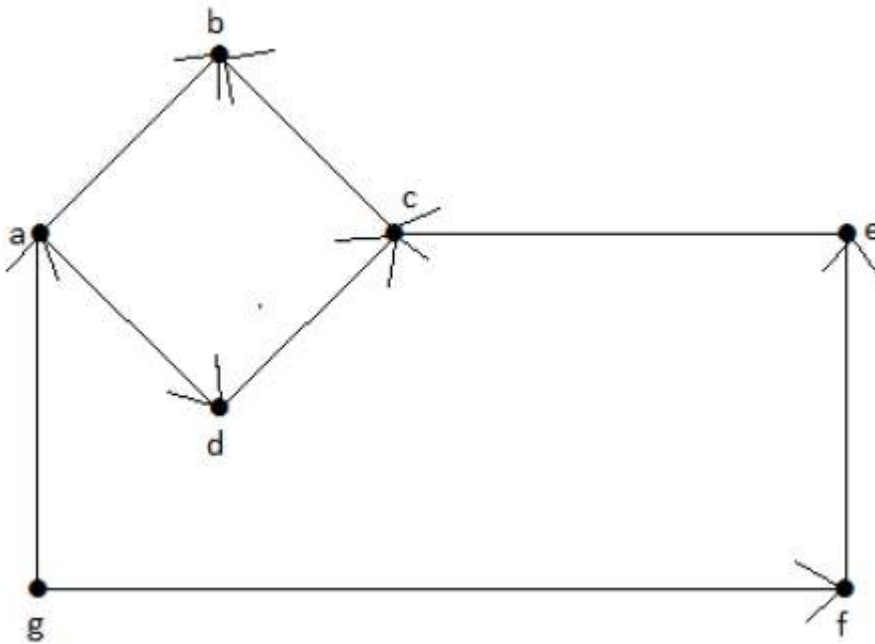
Example

In this graph, 'a', 'b', 'c', 'd', 'e', 'f', 'g' are the vertices, and 'ab', 'bc', 'cd', 'da', 'ag', 'gf', 'ef' are the edges of the graph. Since it is a non-directed graph, the edges 'ab' and 'ba' are same. Similarly other edges also considered in the same way.

Directed Graph

In a directed graph, each edge has a direction.

Example

In the above graph, we have seven vertices 'a', 'b', 'c', 'd', 'e', 'f', and 'g', and eight edges 'ab', 'cb', 'dc', 'ad', 'ec', 'fe', 'gf', and 'ga'. As it is a directed graph, each edge bears an arrow mark that shows its direction. Note that in a directed graph, 'ab' is different from 'ba'.
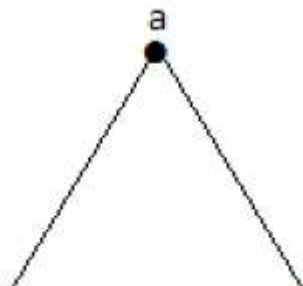
Simple Graph

A graph **with no loops** and **no parallel edges** is called a simple graph.

- The maximum number of edges possible in a single graph with 'n' vertices is $^nC_2$ where $^nC_2 = n(n-1)/2$.

- The number of simple graphs possible with 'n' vertices = $2^{^nC_2} = 2^{n(n-1)/2}$.

Example

In the following graph, there are 3 vertices with 3 edges which is maximum excluding the parallel edges and loops. This can be proved by using the above formulae.
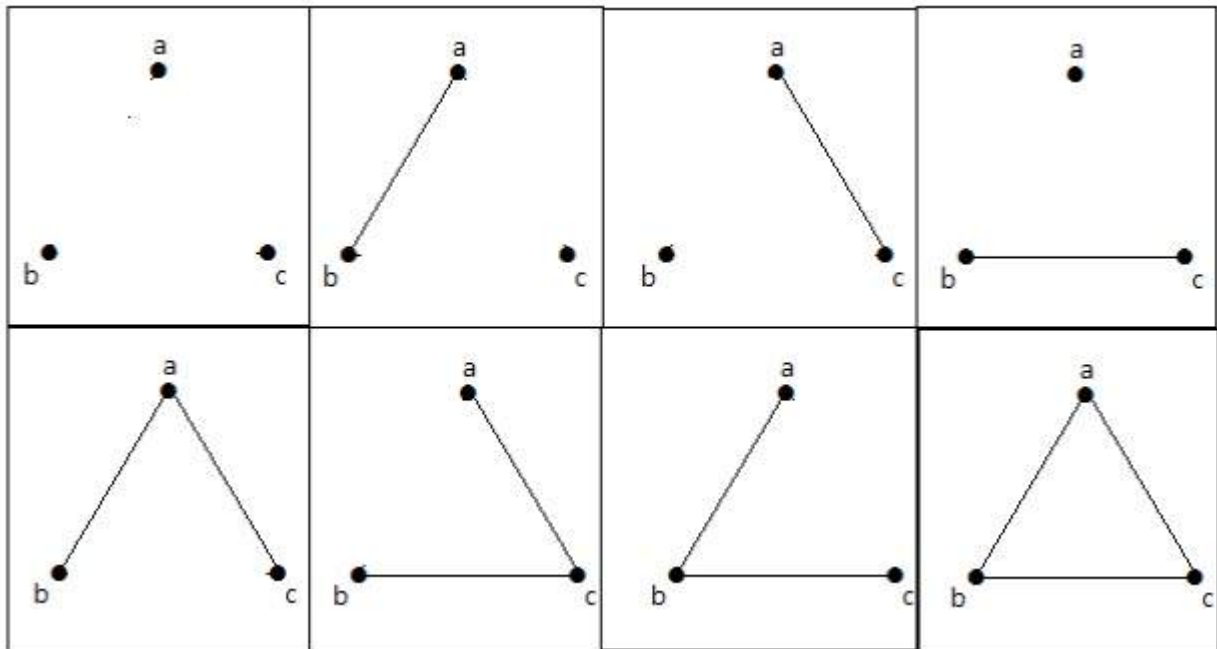
The maximum number of edges with n=3 vertices −

```
nC₂ = n(n–1)/2
    = 3(3–1)/2
    = 6/2
    = 3 edges
```

$$^{n}C_2 = n(n-1)/2 = 3(3-1)/2 = 6/2 = 3 \text{ edges}$$

The maximum number of simple graphs with n=3 vertices −

$$2^{^{n}C_2} = 2^{n(n-1)/2} = 2^{3(3-1)/2} = 2^3 = 8$$

These 8 graphs are as shown below −
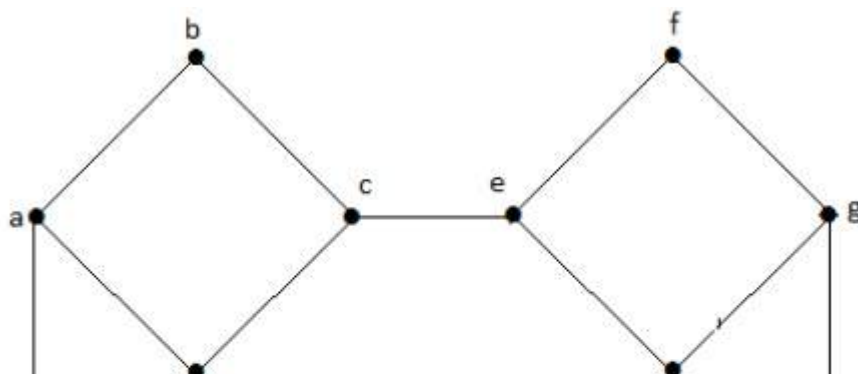

Connected Graph

A graph G is said to be connected **if there exists a path between every pair of vertices**. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.

Example

In the following graph, each vertex has its own edge connected to other edge. Hence it is a connected graph.
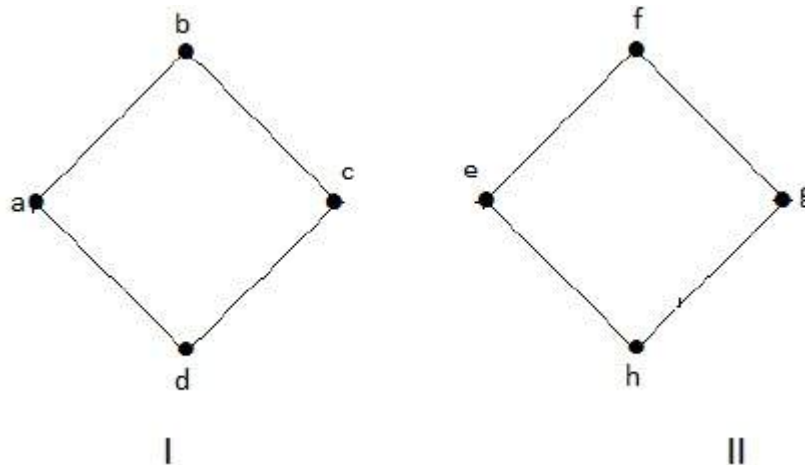
Disconnected Graph

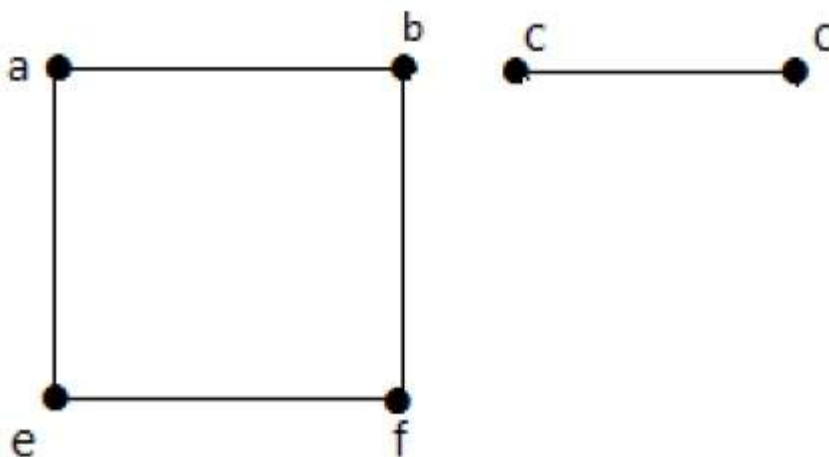A graph G is disconnected, if it does not contain at least two connected vertices.

Example 1

The following graph is an example of a Disconnected Graph, where there are two components, one with 'a', 'b', 'c', 'd' vertices and another with 'e', 'f', 'g', 'h' vertices.

The two components are independent and not connected to each other. Hence it is called disconnected graph.

Example 2

In this example, there are two independent components, a-b-f-e and c-d, which are not connected to each other. Hence this is a disconnected graph.
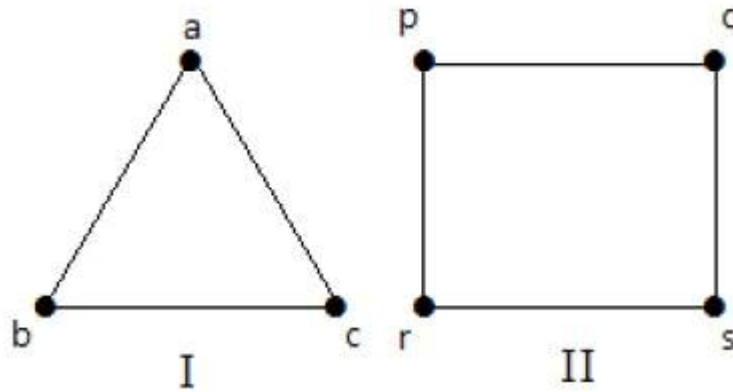
Regular Graph

A graph G is said to be regular, **if all its vertices have the same degree**. In a graph, if the degree of each vertex is 'k', then the graph is called a 'k-regular graph'.

Example

In the following graphs, all the vertices have the same degree. So these graphs are called regular

graphs.

In both the graphs, all the vertices have degree 2. They are called 2-Regular Graphs.

Complete Graph

A simple graph with 'n' mutual vertices is called a complete graph and it is **denoted by 'K$_n$'.** In the graph, **a vertex should have edges with all other vertices,** then it called a complete graph.

In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph.

Example

In the following graphs, each vertex in the graph is connected with all the remaining vertices in the graph except by itself.
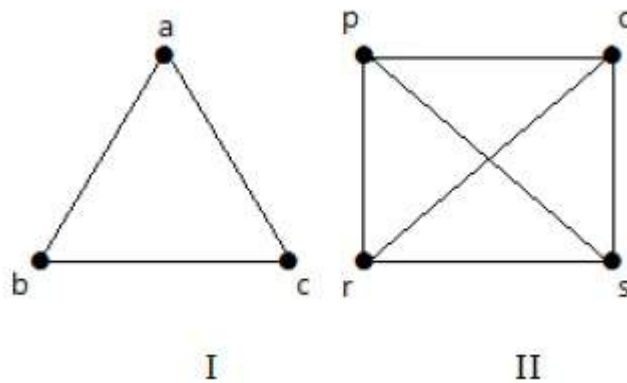
In graph I,

	a	b	c
a	Not Connected	Connected	Connected
b	Connected	Not Connected	Connected
c	Connected	Connected	Not Connected

In graph II,

	p	q	r	s
p	Not Connected	Connected	Connected	Connected

q	Connected	Not Connected	Connected	Connected
r	Connected	Connected	Not Connected	Connected
s	Connected	Connected	Connected	Not Connected

Cycle Graph

A simple graph with 'n' vertices $n >= 3$ and 'n' edges is called a cycle graph if all its edges form a cycle of length 'n'.
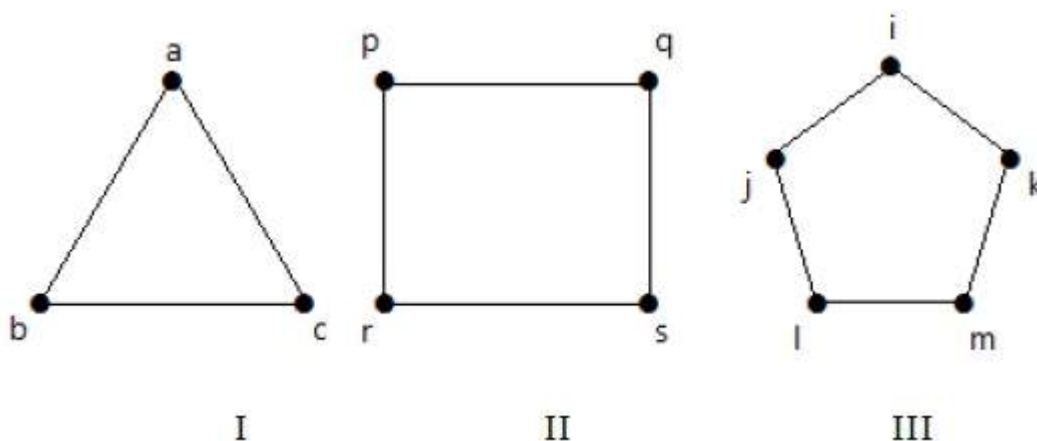
If the **degree of each vertex in the graph is two,** then it is called a Cycle Graph.

Notation – C_n

Example

Take a look at the following graphs –

- Graph I has 3 vertices with 3 edges which is forming a cycle 'ab-bc-ca'.

- Graph II has 4 vertices with 4 edges which is forming a cycle 'pq-qs-sr-rp'.

- Graph III has 5 vertices with 5 edges which is forming a cycle 'ik-km-ml-lj-ji'.

Hence all the given graphs are cycle graphs.

Wheel Graph

A wheel graph is obtained from a cycle graph C_{n-1} by adding a new vertex. That new vertex is called a **Hub** which is connected to all the vertices of C_n.
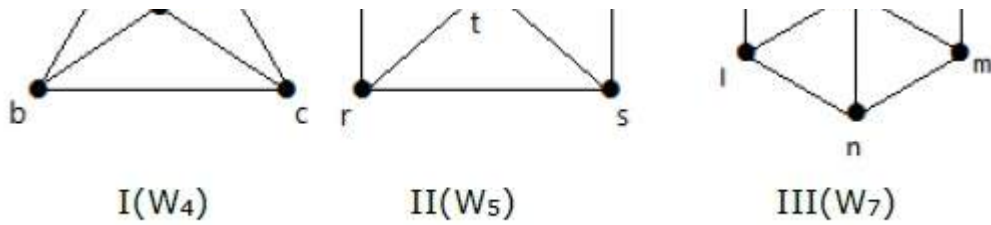
Notation – W_n

```
No. of edges in Wₙ = No. of edges from hub to all other vertices &plus;
                     No. of edges from all other nodes in cycle graph without a hub.
                   = (n−1) + (n−1)
                   = 2(n−1)
```

Example

Take a look at the following graphs. They are all wheel graphs.

I(W₄)   II(W₅)   III(W₇)

In graph I, it is obtained from C_3 by adding an vertex at the middle named as 'd'. It is denoted as W_4.

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Number of edges in W₄ = 2(n-1) = 2(3) = 6
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In graph II, it is obtained from C_4 by adding a vertex at the middle named as 't'. It is denoted as W_5.

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Number of edges in W₅ = 2(n-1) = 2(4) = 8
```
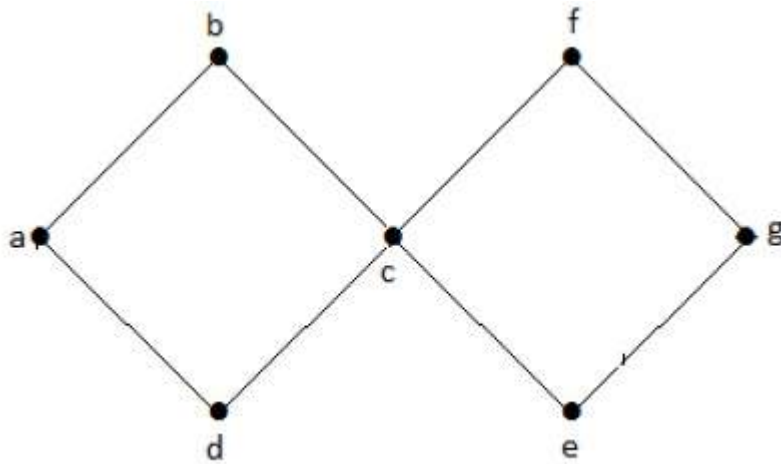
In graph III, it is obtained from C_6 by adding a vertex at the middle named as 'o'. It is denoted as W_7.

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Number of edges in W₄ = 2(n-1) = 2(6) = 12
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Cyclic Graph

A graph **with at least one** cycle is called a cyclic graph.

Example

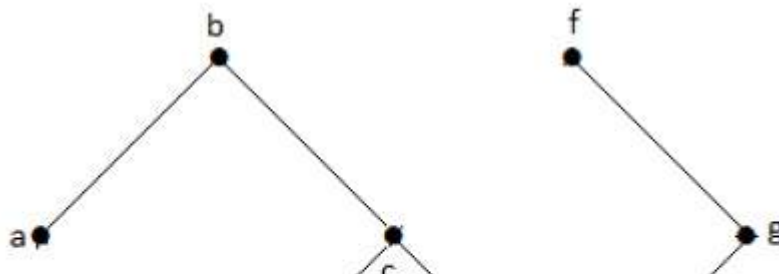

In the above example graph, we have two cycles a-b-c-d-a and c-f-g-e-c. Hence it is called a cyclic graph.

Acyclic Graph

A graph **with no cycles** is called an acyclic graph.

Example

In the above example graph, we do not have any cycles. Hence it is a non-cyclic graph.

Bipartite Graph

A simple graph G = V, E with vertex partition V = {V_1, V_2} is called a bipartite graph **if every edge of E joins a vertex in V_1 to a vertex in V_2**.

In general, a Bipertite graph has two sets of vertices, let us say, V_1 and V_2, and if an edge is drawn, it should connect any vertex in set V_1 to any vertex in set V_2.

Example
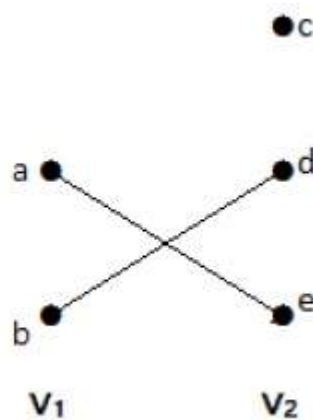
In this graph, you can observe two sets of vertices − V_1 and V_2. Here, two edges named 'ae' and 'bd' are connecting the vertices of two sets V_1 and V_2.
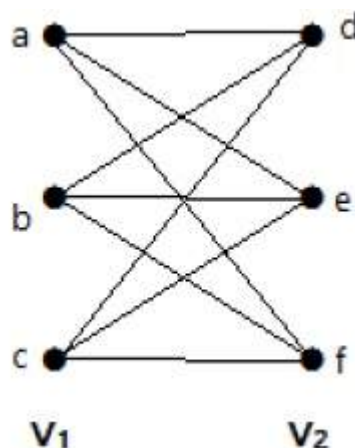
Complete Bipartite Graph

A bipartite graph 'G', G = V, E with partition V = {V_1, V_2} is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2.

In general, a complete bipartite graph connects each vertex from set V_1 to each vertex from set V_2.

Example

The following graph is a complete bipartite graph because it has edges connecting each vertex from set V_1 to each vertex from set V_2.

If $|V_1|$ = m and $|V_2|$ = n, then the complete bipartite graph is denoted by $K_{m,\,n}$.

- $K_{m,n}$ has *m + n* vertices and *mn* edges.

- $K_{m,n}$ is a regular graph if m=n.

In general, **a complete bipartite graph is not a complete graph**.

$K_{m,n}$ is a complete graph if m=n=1.

The maximum number of edges in a bipartite graph with **n** vertices is

$$\lfloor n^2 / 4 \rfloor$$

If n=10, k5, 5= $\lfloor n^2 / 4 \rfloor$ = $\lfloor 10^2 / 4 \rfloor$ = 25

Similarly K6, 4=24

   K7, 3=21

   K8, 2=16

   K9, 1=9

If n=9, k5, 4 = $\lfloor n^2 / 4 \rfloor$ = $\lfloor 9^2 / 4 \rfloor$ = 20
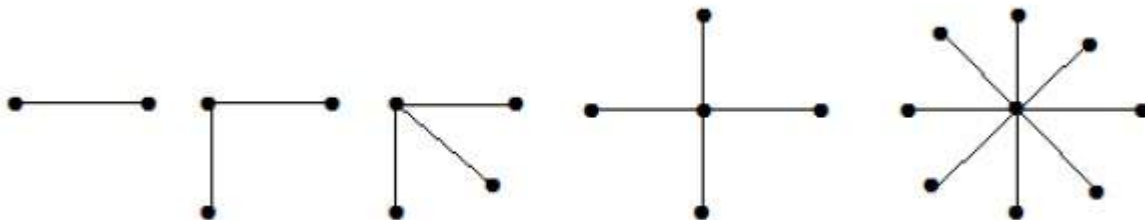
Similarly K6, 3=18

   K7, 2=14

   K8, 1=8

'G' is a bipartite graph if 'G' has no cycles of odd length. A special case of bipartite graph is a **star graph**.

Star Graph

A complete bipartite graph of the form $K_{1,\,n-1}$ is a star graph with n-vertices. A star graph is a complete bipartite graph if a single vertex belongs to one set and all the remaining vertices belong to the other set.

Example

In the above graphs, out of 'n' vertices, all the 'n–1' vertices are connected to a single vertex. Hence it is in the form of $K_{1,\,n-1}$ which are star graphs.
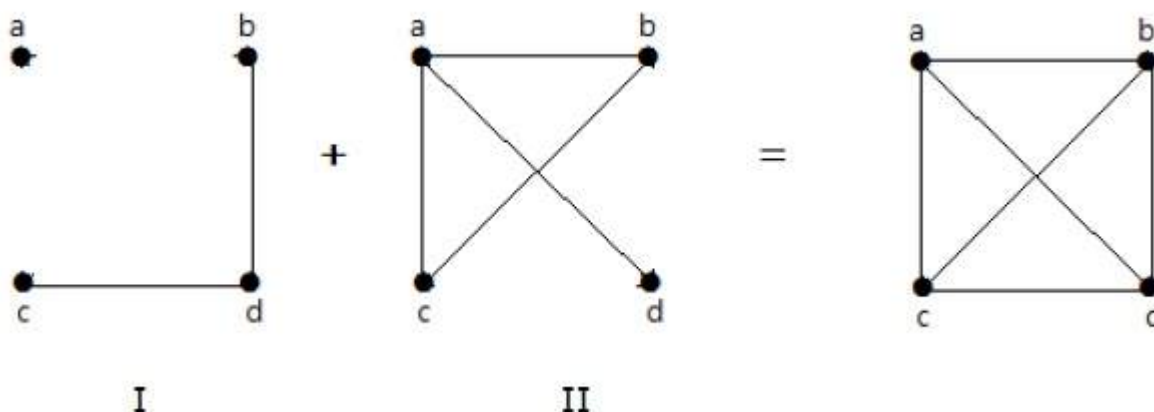
Complement of a Graph

Let '−' if the two vertices are not adjacent in G.

If the edges that exist in graph I are absent in another graph II, and if both graph I and graph II are combined together to form a complete graph, then graph I and graph II are called complements of

each other.

Example

In the following example, graph-I has two edges 'cd' and 'bd'. Its complement graph-II has four edges.

Note that the edges in graph-I are not present in graph-II and vice versa. Hence, the combination of both the graphs gives a complete graph of 'n' vertices.

Note − A combination of two complementary graphs gives a complete graph.

If 'G' is any simple graph, then

$|EG| + |E('-')| = |E(K_n)|$, where n = number of vertices in the graph.

Example

Let 'G' be a simple graph with nine vertices and twelve edges, find the number of edges in '-'.

You have, $|EG| + |E('-')| = |E(K_n)|$

12 + |E('

$99 - 1 / 2 = {}^9C_2$

12 + $|E('-')| = 36$

$|E('-')| = 24$

'G' is a simple graph with 40 edges and its complement '−'.

Let the number of vertices in the graph be 'n'.

We have, $|EG| + |E('-')| = |E(K_n)|$

40 + $38 = nn - 1 / 2$

$156 = nn - 1$

$1312 = nn - 1$

n = 13